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Numerical Methods in Magnetohydrodynamics: A Close Look at Solvers

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ABSTRACT

This paper discusses the critical role of numerical solvers in ideal Magnetohydrodynamics (MHD), covering the purpose, implementation, and interplay of reconstruction methods, Riemann solvers, and the update step. We delve into the nuances of these elements, focusing on Piecewise Parabolic Method (PPM) for the reconstruction, and Godunov's method for the Riemann solver as primary examples.

Keywords: Numerical Methods, Magnetohydrodynamics, PPM, Riemann Solvers

1. INTRODUCTION

Numerical solvers are the cornerstone of computational simulations in various fields of physics, including fluid dynamics and magnetohydrodynamics (MHD). These solvers allow us to approximate solutions to complex differential equations that model real-world phenomena, which are often unsolvable analytically.

In the field of astrophysics, ideal MHD forms the basis for understanding the behaviour of plasma in celestial bodies and the interstellar medium. MHD is a fluid description of magnetized plasmas that combines the principles of fluid dynamics and Maxwell's equations of electromagnetism. Ideal MHD, an approximation of MHD, assumes that the plasma perfectly conducts electricity, i.e., the magnetic field lines are 'frozen' into the fluid. However, the equations of ideal MHD are non-linear partial differential equations and are challenging to solve analytically, except for a limited number of idealized scenarios. Here is where numerical solvers come into play. Numerical solvers allow us to approximate the solutions of these equations on a discretized grid or set of points in space, thereby enabling us to model and predict the behaviour of the plasma under various conditions.

Numerical methods for ideal MHD consist of three main steps: reconstruction, solution of the Riemann problem, and update. The reconstruction step prepares the initial conditions for the Riemann problem by estimating the fluid state at the cell interfaces from the cell-averaged values. The Riemann problem is then solved at each cell interface, yielding the fluxes needed for the update step. Finally, the update step evolves the solution forward in time using these fluxes.

In the following sections, we will delve into these three steps in more detail, discussing their purpose, their common implementations, and the challenges they present.

2. IDEAL MHD EQUATIONS

The ideal MHD equations describe the evolution of fluid variables - density (ρ), velocity ($\vec{v}$), pressure (p), and magnetic field ($\vec{B}$) - in a magnetized plasma. These variables are governed by the following set of conservation equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\frac{\partial (\rho \vec{v})}{\partial t} + \nabla \cdot \left(\rho \vec{v}\vec{v} + p\mathbf{I} - \frac{\vec{B}\vec{B}}{4\pi} \right) = 0 \tag{2}$$

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$$\frac{\partial E}{\partial t} + \nabla \cdot \left[\left(E + p - \frac{|\vec{B}|^2}{8\pi} \right) \vec{v} + \frac{(\vec{v} \cdot \vec{B})\vec{B}}{4\pi} \right] = 0 \tag{3}$$

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) = 0 \tag{4}$$

where $\mathbf{I}$ is the identity matrix, E is the total energy density given by

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\vec{v}|^2 + \frac{|\vec{B}|^2}{8\pi} \tag{5}$$

and γ is the ratio of specific heats. These equations reflect the conservation of mass, momentum, energy, and magnetic flux, respectively.

3. NUMERICAL SOLUTION OF IDEAL MHD EQUATIONS

In the real world, these fluid variables are functions of continuous spatial and temporal variables. However, in a numerical simulation, we approximate these variables at discrete points in space (the grid points) and time. The grid can be one-, two-, or three-dimensional, depending on the problem at hand.

The fundamental strategy of these numerical solvers is to approximate the derivatives in the MHD equations using differences between the values at these discrete grid points. This approach transforms the continuous differential equations into a set of algebraic equations that can be solved on a computer. However, this discretization introduces errors in the solution, and much of numerical analysis is concerned with controlling these errors to obtain an accurate and stable solution.

The key steps in most numerical MHD solvers are reconstruction, Riemann solving, and update. In the reconstruction step, the solver estimates the values of the fluid variables at the interfaces between grid cells from the cell-averaged values. These reconstructed values are then used as input for the Riemann problem, which calculates the fluxes across the cell interfaces. Finally, in the update step, the solver evolves the cell-averaged values forward in time using these fluxes.

In the following sections, we will discuss these steps in detail, focusing on the Piecewise Parabolic Method (PPM) for reconstruction and Godunov's method for the Riemann solver as primary examples.

4. PIECEWISE PARABOLIC METHOD (PPM)

The Piecewise Parabolic Method (PPM) is a high-resolution scheme for numerically solving conservation laws. Introduced by Colella & Woodward (1984), it is designed to avoid the nonphysical oscillations near discontinuities that are seen in lower-order methods.

The PPM reconstruction algorithm can be divided into three major steps:

4.1. *Piecewise Constant Reconstruction*

First, we start with a piecewise constant representation of the fluid variables. Given a grid with cell centers at x_i, we represent the fluid variable $q(x)$ as a constant q_i in each cell $[x_{i-1/2}, x_{i+1/2}]$, where $x_{i\pm1/2}$ are the cell interfaces.

4.2. *Piecewise Linear Reconstruction*

Next, we construct a piecewise linear function that passes through the points (x_{i-1}, q_{i-1}), (x_i, q_i), and (x_{i+1}, q_{i+1}). This linear function is given by

$$q(x) = q_i + \delta q_i \left(\frac{x - x_i}{\Delta x} \right) \tag{6}$$

where Δx is the cell size and δq_i is the slope of the linear function in the ith cell. The slope is computed by a "limited" slope estimate that avoids nonphysical oscillations. A common choice is the minmod limiter:

$$\delta q_i = \text{minmod}\left(\frac{q_{i+1} - q_i}{\Delta x}, \frac{q_i - q_{i-1}}{\Delta x}\right) \tag{7}$$

where the minmod function is defined as

$$\text{minmod}(a, b) = \frac{1}{2}(|a| + |b|)\frac{ab}{|ab|} \tag{8}$$

4.3. *Piecewise Parabolic Reconstruction*

Finally, we modify the linear function to a parabolic function in each cell to better capture the variation of the fluid variables. This parabola is chosen to match the cell-averaged value and the values at the cell interfaces from the linear reconstruction.

The result of this process is a high-order accurate approximation of the fluid variables at the cell interfaces, which are then used as input for the Riemann solver.

5. RIEMANN SOLVER

After the reconstruction step, we have a piecewise function that approximates the fluid variables at the cell interfaces. The next step in the numerical solution of the MHD equations is to solve the Riemann problem at each of these interfaces.

The Riemann problem is an initial value problem with a discontinuity at a given point. In our case, the discontinuity is at the cell interface and the initial values are the left and right states from the reconstruction step. Formally, the Riemann problem for the MHD equations can be written as

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0, \tag{9}$$

$$\mathbf{U}(x, 0) = \{\ U_L \text{ if } x < 0,\ \mathbf{U}_R \text{ if } x > 0 \tag{10}$$

where $\mathbf{U}$ is the vector of conserved variables and $\mathbf{F}(\mathbf{U})$ is the corresponding flux vector.

The solution to the Riemann problem consists of waves propagating away from the initial discontinuity and constant states between these waves. The number, type, and speed of the waves depend on the initial states $\mathbf{U}_L$ and $\mathbf{U}_R$.

In MHD, the Riemann problem is more complex than in hydrodynamics due to the presence of both hydrodynamic and magnetic waves. The solution involves up to seven waves: two fast magnetosonic waves, two slow magnetosonic waves, two Alfven waves, and one contact (entropy) wave.

The Riemann solver computes the intercell fluxes based on the solution to the Riemann problem. These fluxes are then used in the update step to advance the solution in time.

There are many types of Riemann solvers, from exact solvers that find the solution by solving a system of nonlinear equations, to approximate solvers that make certain simplifications to reduce computational cost. A popular choice in MHD is the HLLD solver (Miyoshi & Kusano, 2005), which is an approximate solver that includes all seven MHD waves.

6. UPDATE STEP

The final step in the numerical solution of the ideal MHD equations is the update step. Here, we use the fluxes computed in the Riemann solver to advance the solution from time t^n to time t^{n+1}.

In the context of a finite volume method, the update step is typically done using a conservative scheme, such as the Euler method, which can be expressed as follows,

$$\mathbf{U}i^{n+1} = \mathbf{U}i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{F}i + 1/2^n - \mathbf{F}i - 1/2^n\right), \tag{11}$$

where $\mathbf{U}i^n$ is the cell-averaged value of the conserved variables in cell i at time n, $\mathbf{F}i + 1/2^n$ and $\mathbf{F}^n_{i-1/2}$ are the fluxes at the cell interfaces computed by the Riemann solver, and Δt and Δx are the time step and cell size, respectively.

This scheme conserves the quantities integrated over each cell, provided that the fluxes are consistent with the conservation laws. It is also first-order accurate in time, meaning that the local truncation error is proportional to Δt. Higher-order accuracy in time can be achieved by using a more advanced time-stepping scheme, such as a Runge-Kutta method.

After the update step, the cycle of reconstruction, Riemann solve, and update is repeated until the solution has been advanced to the desired final time. This iterative procedure forms the backbone of most numerical MHD codes.

7. REFERENCES

REFERENCES

Balsara, D. S. 1998, ApJS, 116, 133

Butler, E. M., Hawley, J. F., Toth, G. 2010, ApJ, 709, 1309

Colella, P., Woodward, P. R. 1984, Journal of Computational Physics, 54, 174-201

Toro, E. F. 2009, Riemann Solvers and Numerical Methods for Fluid Dynamics, 3rd edn. (Springer)

Miyoshi, T., Kusano, K. 2005, Journal of Computational Physics, 208, 315–344

Stone, J. M., Gardiner, T. A., Teuben, P., Hawley, J. F., Simon, J. B. 2008, ApJS, 178, 137